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FERMAT'S METHOD OF INFINITE DESCENT.

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The method of infinite descent and the method of mathematical induction, which may appropriately be called the method of infinite ascent, both came into use in the seventeenth century in that productive period made famous by Descartes, Cavalieri, Pascal, Wallis and Fermat. Mathematical induction was first brought into prominence by Pascal, although it was invented by Maurolycus[1] and used in his arithmetic which was published in 1575. It is a method of infinite ascent from a special case, for which the theorem to be proved is verified by means of the argument from n to $n + 1$. The method of infinite descent, which was invented by Fermat, is not so widely known. Fermat's own account of it is to be found in a letter entitled "*Relation des nouvelles découvertes en la science des nombres*" which he wrote to Pierre de Carcavi in 1659.[2] In this letter he tells Carcavi that he has discovered a new method of demonstration and applied it successfully to the solution of a considerable number of problems in the theory of numbers. He calls it the method of infinite or unlimited descent (descente infinie ou indéfinie) and says that at first he applied it only to negative propositions such as these two:

"*No number of the form* $3k - 1$ *can be of the form* $x^2 + 3y^2$."

"*There is no right triangle whose sides are integers whose area is equal to the square of an integer.*"

He then gives the following abstract of his proof of the latter: "The proof is made by *ἀπαγωγήν εις ἀδυνατον*[3] in this manner: If there is a right triangle with integral sides and with an area equal to the square of an integer, then there is a second triangle, smaller than the first, which has the same property; and if there is a second, smaller than the first, which has the same property, then there is, by like reasoning, a third smaller than the second; and then a fourth, a fifth, and so on ad infinitum. But there is not an infinite number of integers less than a given integer. From which one concludes that it is impossible that there should be a right triangle with integral sides and with an area which is the square of an integer." In the next paragraph Fermat says: "It was a long time before I was able to apply my method to affirmative questions, because the way and manner of getting at them is much more difficult than that which I employ with negative theorems. So much so that, when I had to prove that every prime number of the form $4k + 1$ is made up of two squares, I found myself in much torment. But at last a certain reflection many times repeated gave me the necessary light, and affirmative questions yielded to my method with some new principles by which sheer necessity compelled me to supplement it. This development of my argument in the case of affirmative questions takes the following line. If a prime number of the form $4k + 1$ selected at random is not made up of two squares, there will exist another prime number of the same sort but less than the given number, and again a third still smaller and so on descending ad infinitum until one comes to the number 5, which is the smallest of all numbers of the kind in question and which the argument would require not to be made up of two squares, although in fact it is so made up. From which one must infer, by reductio ad absurdum, that all numbers of the kind in question are in consequence made up of two squares."

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[1] W. H. Bussey, "The Origin of Mathematical Induction," this MONTHLY, vol. 24, pp. 199–207.

[2] *Oeuvres de Fermat*, Vol. 2, pp. 431–436.

[3] Greek for *reductio ad absurdum*.

The foregoing is a good account of Fermat's method although it leaves out all details of the proof. Fermat's proof, like so many others of his, is not extant. The proof was actually made by Euler.[1]

The American college student learns the method of mathematical induction by means of theorems on the summation of such series as

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \tfrac{1}{4}n^2(n + 1)^2$$

and of the theorems about the algebraic divisibility of $x^n \pm y^n$ by $x \pm y$. The proof involves the verification of the theorem in a special case and the ascent argument from n to $n + 1$. But the proof can be made just as easily by means of the verification in a special case and a descent argument from n to $n - 1$. For example the theorem that $x^n - y^n$ is divisible by $x - y$ for all integral values of n may be proved as follows: Suppose that there is an integer n for which the theorem is not true. Then the identity

$$\frac{x^n - y^n}{x - y} \equiv x^{n-1} + \left(\frac{x^{n-1} - y^{n-1}}{x - y}\right) y,$$

obtained by ordinary long division, proves that $x^{n-1} - y^{n-1}$ is not divisible by

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[1] For a proof of this theorem, in a form which is a modification of the proof described by Fermat, see H. WEBER and J. WELLSTEIN, *Encyclopädie der Elementar-Mathematik*, Vol. 1, p. 285.

$x - y$; that is, the theorem is not true for $n - 1$. This descent argument, repeated a finite number of times, requires that the theorem be not true when $n = 2$; in other words, that $x^2 - y^2$ be not divisible by $x - y$. This furnishes the contradiction needed to complete the proof by reductio ad absurdum. The descent is not an infinite descent but rather a finite descent to the special verified case of the theorem. It is really a proof by mathematical induction turned around. Any proof by mathematical induction can be put in this descent form. The proof is well characterized by the description which Fermat gave of his method in connection with the theorem that every prime of the form $4k + 1$ equals the sum of two squares. But Fermat's proof of this theorem does not involve an argument from n to $n - 1$. Nor is the descent in this case one from a prime number to the next smaller prime number. The essential feature of his method is not any such regular descent as that given by the argument from n to $n - 1$. It is rather a descent from an integer n involved in the theorem to some smaller integer; just how much smaller may not be known. This is very well illustrated by the proof that $x^4 + y^4 = z^4$ has no solution in integers different from zero. The proof in outline is this: If $x^4 + y^4 = z^4$ has a solution in such integers, so has $x^4 + y^4 = z^2$; if there exists a solution (x_1, y_1, z_1) of this equation, there exists another solution (x_2, y_2, z_2) in which z_2 is an integer less than z_1; and by the same argument another solution (x_3, y_3, z_3) in which z_3 is less than z_2; and so forth ad infinitum. The contradiction needed for the completion of the argument by reductio ad absurdum is furnished by the fact that there is not an infinite number of positive integers less than a given one. But the contradiction may just as well be obtained from the special case $x^4 + y^4 = 1^2$, which by inspection has no solution in integers different from zero. The descent argument that $z_2 < z_1$ holds without exception. But in the proof that $x^n - y^n$ is divisible by $x - y$ the descent argument fails when $n = 1$. The fact that it fails makes the proof a finite descent proof rather than an infinite descent proof. But it should be said in this connection that the proof that $x^4 + y^4 = z^2$ has no integral solution may be thought of as a finite descent to the special case $x^4 + y^4 = 1^2$ mentioned above.

The proof that $x^4 + y^4 = z^4$ has no integral solution may also be put in this form.[1] Let it be assumed that $x^4 + y^4 = z^2$ has one or more solutions and let (x_1, y_1, z_1) be a solution in which z_1 is the smallest possible. Then the descent argument proves that there is a solution (x_2, y_2, z_2) in which z_2 is less than z_1. The required contradiction is furnished by the fact that there is thus proved to be a smaller than the smallest. In any true infinite descent proof, that is in a proof in which the descent argument holds without exception, the contradiction needed for the reductio ad absurdum argument may be obtained in any one of the three ways mentioned, namely from the fact that there is not an infinite number of integers less than a given integer, or from the descent to a special verified case of the theorem, or from the fact that there is not a smaller than the smallest. But when the descent argument does not hold without exception, as in the proof that $x^n - y^n$ is divisible by $x - y$, the contradiction must be obtained from the descent to a special verified case of the theorem. It is this finite descent proof which is mathematical induction turned around. In thinking of these two types of descent proof one wonders if there is not another type of ascent proof than the regular ascent from n to $n + 1$ of mathematical induction. The well-known proof, due to Euclid, that there is no greatest prime is one such. The argument is that if p is any prime, the number $N = p! + 1$ is a greater prime or else one of its prime factors is; that is, if p is any prime, there is a greater prime.

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[1] See H. WEBER and J. WELLSTEIN, *l. c.*, p. 284.

It is hard to account for Fermat's trouble in applying his method to positive or affirmative propositions. His method is to assume the opposite of what he wants to prove and to show that the assumption leads to a contradiction. Sometimes the statement of a proof by reductio ad absurdum is awkward, but it may be just as awkward for a negative proposition as for an affirmative one. His difficulty reminds one of the student who can prove by mathematical induction that $x^{2n} - y^{2n}$ is divisible by $x + y$ but is at a loss to know how to use the same method to prove that $x^{2n} + y^{2n}$ is not divisible by $x + y$.

Fermat, in his letter to Carcavi, says or implies that his method of infinite descent is applicable to the following propositions, some of them having already been mentioned in this paper:

1. There is no number of the form $3k - 1$ which is also of the form $x^2 + 3y^2$.
2. There is no right triangle whose sides are integers and whose area equals a square number.
3. Every prime of the form $4k + 1$ is equal to the sum of two squares.
4. Every number is either a square or the sum of two, three or four squares.
5. $x^2 - ay^2 = 1$ has an infinite number of integral solutions if a is not a square.
6. There is no integral solution of $x^3 + y^3 = z^3$.
7. There is one and only one integral solution of $x^2 + 2 = y^3$, namely, $x = 5$, $y = 3$.
8. There are two and only two integral solutions of $x^2 + 4 = y^3$, namely, $x = 2$, $y = 2$ and $x = 11$, $y = 5$.
9. Every number of the form $2^{2^n} + 1$ is a prime number.
10. There are only two integral solutions of $(2x^2 - 1)^2 = 2y^2 - 1$, namely, $x = 1$, $y = 1$ and $x = 2$, $y = 5$.

Fermat does not say explicitly that he succeeded in proving all of these propositions but his language gives one that impression. He does say that he proved Nos. 1, 2, 3, 4, 5, 10. Of Nos. 6, 7, 8, 9 he says: "I afterwards considered certain propositions which, although negative, present very great difficulty, the method of applying the descent being altogether different from that of the preceding cases." Of No. 9 he says further: "This is a very subtle and very ingenious research and, although it is stated affirmatively, it is negative, since to say that a number is prime is the same as to say that it cannot be divided by any number."

This Proposition No. 9 is not true. For when $n = 5$ the number

$$2^{2^n} + 1 = 4{,}294{,}967{,}297 = 641 \cdot 6{,}700{,}417.$$[1]

It is often stated that Fermat did not claim to have proved this theorem. This statement is based on the fact that in a letter to Pascal[2] in August, 1654, he says that, although he is convinced of its truth, he has not been able to find a valid demonstration; and that in a letter to Digby[3] in June, 1658, he says that he is seeking a demonstration of this beautiful theorem. But the quotation given above is from the 1659 letter to Carcavi and in it Fermat seems to say that he has proved it by the method of infinite descent.

For a brief account of Fermat's method of infinite descent, especially of its relation to the solution of Diophantine equations, the reader is referred to R. D. CARMICHAEL'S "*Diophantine Analysis,*" pp. 14–21. The reader is also referred to the supplement of Sir THOMAS L. HEATH'S "*Diophantus of Alexandria,*" Cambridge University Press, 2d edition, 1910, which contains a number of references to the method of infinite descent.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

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In a previous paper[4] I showed how the study of the line in analytic geometry could be presented from the viewpoint of functions and relations between variables. The present article explains how logarithms and exponential relations are introduced on the same basis.

I begin the work by solving some problem leading to an exponential relation between two variables. Most such problems taken from physics or chemistry lead to a negative exponent, which is undesirable at the beginning of the work. However, a suitable problem is that of determining the number N of bacteria in a solution at a time t, knowing the number N_0 present at $t = 0$ and knowing the rate of increase. The conditions are similar to those for finding the length of a rod with increasing temperature, a problem studied under linear functions, but different in that the coefficient of expansion of the rod is independent of the temperature while the bacteria increase at a rate dependent upon t. If the bacteria increase r per cent. in a unit of time, then

$$N = N_0\left(1 + \frac{r}{100}\right)^t.$$

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[1] This was proved by Euler in 1732. For this and other comments on the theorem see W. W. R. Ball, *Mathematical Recreations and Essays*, 5th edition, p. 39. Also see R. C. Archibald's *Remarks on "Klein's famous problems of elementary geometry,"* this MONTHLY, Vol. 21, p. 248.

[2] *Oeuvres de Fermat*, Vol. 2, p. 307.

[3] *Ibid.*, p. 402.

[4] "The Linear Function and the Line," in this MONTHLY, Nov., 1917, page 406. This was the third of a series of related papers of which the other two are:
"The Unification of Freshman Mathematics," April, 1916, page 101;
"The Presentation of the Notion of Function," Sept., 1917, p. 309.